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Multi-particle scattering at the same position in the Bethe ansatz for two-band electrons with spin-exchange interaction

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Abstract

The same-position scattering (SPS) of more than two electrons in a one-dimensional model of two-band electrons with spin-exchange interaction is discussed. The boundary conditions of three- and four-particle SPS are given. It is shown that the conditions can be fulfilled by the two-particle boundary conditions for the Bethe ansatz (BA) wavefunction. Consequently, the definition of the BA wavefunction can be extended to those cases of more than two particles occupying the same position. Therefore, unlike the case in lattice models in which configurations with more than two particles at one site are excluded in applying the approach, the BA is valid without the exclusion of multi-particle SPS in the spin-exchange model. A relation between the $SU(2) \times SU(2)$ symmetry and the BA equation is also indicated.

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1. Introduction

Recently considerable attention has been directed to the strongly correlated electrons with an orbital degree of freedom due to the experimental progress related to 3d electrons [1]. For one-dimensional single-band electrons, the Bethe ansatz (BA) approach has been successfully applied to the δ-function interacting fermions [2] and the Hubbard model [3], respectively for the spatially continuous case and the lattice case. There has been much theoretical work by extending the BA to the degenerate systems [4–12]. As the single-band electron possesses only two internal degrees of freedom, it does not involve the scattering of more than two electrons at the same position when applying the BA. Whereas the existence of the orbital degree of freedom increases the internal components of the electrons, and as the Pauli principle only prohibits the spatially double occupation for electrons with the same component, the same-position scattering (SPS) of n $(n > 2)$ electrons with different components naturally should be taken into account in degenerate electronic systems, but the original approach of the BA only involves the two-particle scattering, and the factorization of a multi-particle matrix into

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products of two-particle S-matrices applies only to the *transfer* matrices which transfer the amplitudes between different region sectors, not referring to the *same-position* multi-particle scattering. In other words, the multi-particle SPS is not under the consideration of the original BA. As a result, extending the BA to degenerate electron systems will encounter the problem of consistence between the BA and the multi-particle scattering. If the BA wavefunction does not fulfill the conditions required by the multi-particle SPS, it will not be the solution of the degenerate system. This has been realized in the degenerate Hubbard model for which the direct $SU(n)$ extension of the $SU(2)$ BA does not solve the configurations with multiply occupied sites [6]. As a matter of fact, the theoretical works in the BA approach have been based on the exclusion of more than two electrons at the same position [6–10].

In this paper we shall consider a one-dimensional system of two-band electrons with spin-exchange interaction [11, 12]. We work out the three- and four-electron SPS conditions and show that they can be fulfilled by the two-electron SPS equation in the BA wavefunction. Therefore we conclude that the BA wavefunction can be exactly extended to multi-electron SPS cases and the BA is exactly valid for the model.

2. The model and its symmetry

The Hamiltonian under consideration reads

$$H = \frac{1}{2}\sum_{m,\sigma}\int \mathrm{d}x\,\frac{\partial}{\partial x}\,C^{\dagger}_{m\sigma}(x)\frac{\partial}{\partial x}C_{m\sigma}(x) + \frac{c}{2}\sum_{mm'\sigma\sigma'}\int \mathrm{d}x\int \mathrm{d}x'\,\delta(x-x')C^{\dagger}_{m\sigma}(x)C^{\dagger}_{m'\sigma'}(x')C_{m'\sigma}(x')C_{m\sigma'}(x) \tag{1}$$

where $C^{\dagger}_{m\sigma}(x)$ creates an electron at site x with spin component σ and band label m ($m = 1, 2$). Because of the *spin-exchange* interaction, the system has a $SU(2)\times SU(2)$ symmetry generated by

$$\begin{aligned}
S^z &= \tfrac{1}{2}\sum_m \int \mathrm{d}x\,\left[C^{\dagger}_{m\uparrow}(x)C_{m\uparrow}(x) - C^{\dagger}_{m\downarrow}(x)C_{m\downarrow}(x)\right] \\
S^+ &= \sum_m \int \mathrm{d}x\,C^{\dagger}_{m\uparrow}(x)C_{m\downarrow}(x) \qquad S^- = \sum_m \int \mathrm{d}x\,C^{\dagger}_{m\downarrow}(x)C_{m\uparrow}(x) \\
T^z &= \tfrac{1}{2}\sum_\sigma \int \mathrm{d}x\,\left[C^{\dagger}_{2\sigma}(x)C_{2\sigma}(x) - C^{\dagger}_{1\sigma}(x)C_{1\sigma}(x)\right] \\
T^+ &= \sum_\sigma \int \mathrm{d}x\,C^{\dagger}_{2\sigma}(x)C_{1\sigma}(x) \qquad T^- = \sum_\sigma \int \mathrm{d}x\,C^{\dagger}_{1\sigma}(x)C_{2\sigma}(x).
\end{aligned} \tag{2}$$

These operators $\{S^z, S^+, S^-\}$ and $\{T^z, T^+, T^-\}$ obey the $SU(2)$ commutation relations

$$[S^z, S^{\pm}] = \pm S^{\pm} \qquad [S^+, S^-] = 2S^z;$$

likewise for the T operators. All these operators commute with the Hamiltonian so that the system possesses the $SU(2)\times SU(2)$ symmetry.

3. Bethe ansatz and two-particle scattering

In the Hilbert space of N particles, the energy eigenequation in the first quantization is given by

$$\left[-\frac{1}{2}\sum_i^N \frac{\partial^2}{\partial x_i^2} + c\sum_{i<j}\mathcal{P}^{ij}_{\sigma}\delta(x_i - x_j)\right]\psi = E\psi \tag{3}$$

where $\mathcal{P}_\sigma^{ij}$ is the spin-exchange operator of electrons i and j. We adopt the following form of BA wavefunction:

$$\psi^{(Q)}_{m_1\sigma_1\ldots m_N\sigma_N}(x_1,\ldots,x_N)=\sum_{P\in S_N}A^{(Q)}_{m_{P_1}\sigma_{P_1}\ldots m_{P_N}\sigma_{P_N}}\mathrm{e}^{\mathrm{i}\sum_j k_j x_{P_j}} \tag{4}$$

which solves the eigenequation

$$-\frac{1}{2}\sum_i^N\frac{\partial^2}{\partial x_i^2}\psi^{(Q)}=E\psi^{(Q)}$$

in the region $R_Q : x_{Q_1}<\cdots<x_{Q_N}$ with eigenenergy $E=\sum_i^N k_i^2/2$. On the barrier $x_i=x_j$ with the spin-exchange interaction, the connection condition is given by

$$\begin{aligned}&\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\left(\psi^{(Q_1\ldots ji\ldots Q_N)}_{m_1\sigma_1\ldots m_N\sigma_N}\big|_{x_j=x_i+}-\psi^{(Q_1\ldots ij\ldots Q_N)}_{m_1\sigma_1\ldots m_N\sigma_N}\big|_{x_j=x_i-}\right)\\&\qquad=c\mathcal{P}_\sigma^{ij}\left[\psi^{(Q_1\ldots ij\ldots Q_N)}_{m_1\sigma_1\ldots m_N\sigma_N}+\psi^{(Q_1\ldots ji\ldots Q_N)}_{m_1\sigma_1\ldots m_N\sigma_N}\right]_{x_i=x_j}\end{aligned} \tag{5}$$

where $(Q_1\ldots ij\ldots Q_N)$ labels a region with $Q_r=i$ and $Q_{r+1}=j$ for some r. Two adjacent sectors $(Q_1\ldots ij\ldots Q_N)$ and $(Q_1\ldots ji\ldots Q_N)$ abut on the connecting boundary $x_i=x_j$. The two-particle scattering relation (5) is obtained by integrating the eigenequation (3) over a Gaussian box with the hyperplane $x_i=x_j$ cut across. The amplitudes of the same $\exp(\mathrm{i}\sum_j k_j x_{P_j})$ in (5) give the relation

$$\begin{aligned}&(\mathrm{i}k_s-\mathrm{i}k_t)\left[A^{(\ldots ji\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_j\ldots}-A^{(\ldots ji\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}-A^{(\ldots ij\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_j\ldots}+A^{(\ldots ij\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}\right]\\&\qquad=c\mathcal{P}_\sigma^{ij}\left[A^{(\ldots ij\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_j\ldots}+A^{(\ldots ij\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}+A^{(\ldots ji\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_t\ldots}+A^{(\ldots ji\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}\right]\end{aligned} \tag{6}$$

where $P_s=i$ and $P_t=j$ in $A^{(Q_1\ldots ij\ldots Q_N)}_{m_{P_1}\sigma_{P_1}\ldots m_{P_s}\sigma_{P_s}\ldots m_{P_t}\sigma_{P_t}\ldots m_{P_N}\sigma_{P_N}}$. The single-valuedness of the wavefunction on the boundary requires

$$A^{(\ldots ij\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_j\ldots}+A^{(\ldots ij\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}=A^{(\ldots ji\ldots)}_{\ldots m_i\sigma_i\ldots m_j\sigma_t\ldots}+A^{(\ldots ji\ldots)}_{\ldots m_j\sigma_j\ldots m_i\sigma_i\ldots}. \tag{7}$$

From (6) and (7) one can find the two-particle S-matrix

$$\begin{aligned}&S^{ij}=\frac{(k_s-k_t)I_\sigma^{ij}-\mathrm{i}c\mathcal{P}_\sigma^{ij}}{(k_s-k_t)-\mathrm{i}c}\,\frac{(k_s-k_t)I_m^{ij}+\mathrm{i}c\mathcal{P}_m^{ij}}{(k_s-k_t)+\mathrm{i}c}\\&S^{ij}A^{(Q_1\ldots ij\ldots Q_N)}_{m_{P_1}\sigma_{P_1}\ldots m_{P_s}\sigma_{P_s}\ldots m_{P_t}\sigma_{P_t}\ldots m_{P_N}\sigma_{P_N}}=A^{(Q_1\ldots ji\ldots Q_N)}_{m_{P_1}\sigma_{P_1}\ldots m_{P_s}\sigma_{P_s}\ldots m_{P_t}\sigma_{P_t}\ldots m_{P_N}\sigma_{P_N}}\end{aligned} \tag{8}$$

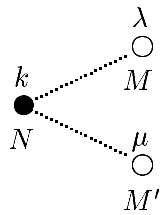
where $\mathcal{P}_\sigma^{ij}$ exchanges the spins of particles i and j in $A^{(Q_1\ldots ij\ldots Q_N)}_{m_{P_1}\sigma_{P_1}\ldots m_{P_s}\sigma_{P_s}\ldots m_{P_t}\sigma_{P_t}\ldots m_{P_N}\sigma_{P_N}}$ while $\mathcal{P}_m^{ij}$ exchanges band indices of the electrons. It should be noted that $\psi^{(Q_1\ldots ij\ldots Q_N)}|_{x_j=x_i+}$ and $\psi^{(Q_1\ldots ji\ldots Q_N)}|_{x_j=x_i-}$ in (5) are respectively in two adjacent sectors $(Q_1\ldots ij\ldots Q_N)$ and $(Q_1\ldots ji\ldots Q_N)$. The transfer matrix for non-adjacent sectors factorizes into products of the two-particle S-matrices. The diagonalization of the N-particle transfer matrices combined with periodic condition leads to the BA equations which determine the spectrum. Therefore, the two-particle scattering boundary condition and the periodic condition are both conditions that the BA is involved. However, the SPS conditions of more than two particles are required for the system since the particles are two-band electrons. These conditions have not been explicitly considered in the original BA, and the consistence between the BA and the additional conditions should be verified to guarantee the exactness of the BA solution.

Before proceeding to the multi-electron SPS, we shall point out an interesting connection between the BA and the Dynkin diagram of D_2 Lie algebra which corresponds to the

$SU(2) \times SU(2)$ symmetry of the present model. The BA equations [11] for the system are written as

$$
\begin{aligned}
e^{ik_j L} &= \prod_{\alpha=1}^{M} \Xi_{1/2}(k_j - \lambda_\alpha) \prod_{\beta=1}^{M'} \Xi_{-1/2}(k_j - \mu_\beta) \\
1 &= -\prod_{j=1}^{N} \Xi_{-1/2}(\lambda_\alpha - k_j) \prod_{\gamma=1}^{M} \Xi_1(\lambda_\alpha - \lambda_\gamma) \\
1 &= -\prod_{j=1}^{N} \Xi_{-1/2}(\mu_\beta - k_j) \prod_{\gamma=1}^{M'} \Xi_1(\mu_\beta - \mu_\gamma)
\end{aligned}
\tag{9}
$$

where $\Xi_l = [x + ilc] / [x - ilc]$ and $\{k_j | j = 1, \ldots, N\}$, $\{\lambda_\alpha | \alpha = 1, \ldots, M\}$ and $\{\mu_\beta | \beta = 1, \ldots, M'\}$ are respectively charge rapidities, spin rapidities and the band rapidities. We write out the BA equation in the form (9) so that it is easy to remember by means of the 'Dynkin diagram' of D_2 Lie algebra



where the dark dot is added to represent the charge rapidity k_j, which also takes an angle of 120° relative to both the simple roots, r_1 and r_2, of D_2 Lie algebra. The subscripts of Ξ in the second and the third equations of (9) are then related to the covariant components of the simple roots when the simple roots and the added vector (dark dot) are chosen as the non-orthogonal basis, respectively, $r_1 = (-1/2,\ 1,\ 0)$, $r_2 = (-1/2,\ 0,\ 1)$. This connection exists because the symmetry of the model is $SU(2) \times SU(2)$, the generators of which constitute D_2 Lie algebra. Such a kind of connection was noticed in the $SU(4)$ Hubbard model [10].

4. Three-particle scattering at the same position

The continuity of the wavefunction of two particles at the same position has been used in the BA; equation (7) is the corresponding amplitude relation. With the help of (7), it is easy to verify the single-valuedness of the wavefunction with three or four electrons at the same position:

$$
\begin{aligned}
&\psi^{(\ldots Q_r Q_{r+1} Q_{r+2} \ldots)}\Big|_{x_{Q_r} = x_{Q_{r+1}} = x_{Q_{r+2}}} = \psi^{(\ldots Q'_r Q'_{r+1} Q'_{r+2} \ldots)}\Big|_{x_{Q'_r} = x_{Q'_{r+1}} = x_{Q'_{r+2}}} \\
&\psi^{(\ldots Q_r Q_{r+1} Q_{r+2} Q_{r+3} \ldots)}\Big|_{x_{Q_r} = x_{Q_{r+1}} = x_{Q_{r+2}} = x_{Q_{r+3}}} = \psi^{(\ldots Q'_r Q'_{r+1} Q'_{r+2} Q'_{r+3} \ldots)}\Big|_{x_{Q'_r} = x_{Q'_{r+1}} = x_{Q'_{r+2}} = x_{Q'_{r+3}}}
\end{aligned}
\tag{10}
$$

where $(Q_r Q_{r+1} Q_{r+2}), (Q'_r Q'_{r+1} Q'_{r+2}) \in S_3 = \{(ijk)\}$ and $(Q_r Q_{r+1} Q_{r+2} Q_{r+3})$, $(Q'_r Q'_{r+1} Q'_{r+2} Q'_{r+3}) \in S_4 = \{(ijkl)\}$ are respectively any permutations of three specific electrons ijk and four electrons $ijkl$.

To obtain the SPS condition of three particles i, j and k we need to integrate the eigenequation (3) over an arbitrarily shaped volume; the intersecting line $x_i = x_j = x_k$ goes through the volume in the subspace of x_i, x_j and x_k. The arbitrarily shaped volume can be cut into an ϵ-sized hexagonal prism of which a cross section is shown in figure 1; the integration of the eigenequation over the part outside the prism automatically cancels due

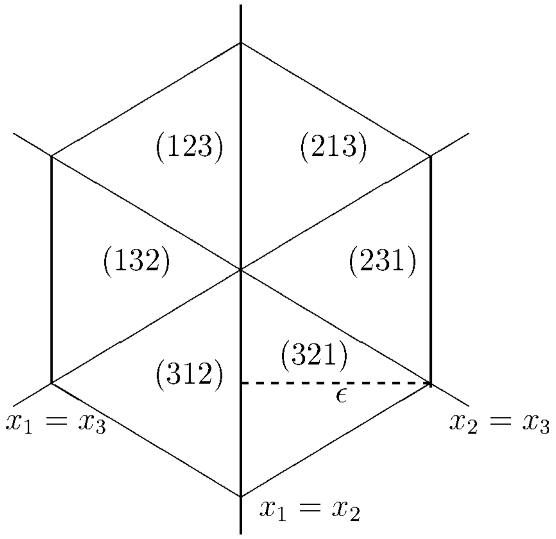


Figure 1. A cross section of the hexagonal prism in the three-dimensional subspace of x_1, x_2 and x_3. The centre axis of the prism is along the intersecting line $x_1 = x_2 = x_3$, and the labels of the three interacting particles ijk are set to be 123.

to (5), so it is sufficient to consider only integration over the prism. Setting the size parameter ϵ to be infinitesimal and keeping the lowest order of ϵ in the expansion of the Gaussian integral, we find the following equations required by three particles at the same position:

$$\begin{aligned}\left(\frac{\partial}{\partial x_1}-\frac{\partial}{\partial x_2}\right)&\left[\psi^{(Q_1\ldots 231\ldots Q_N)}-\psi^{(Q_1\ldots 132\ldots Q_N)}\right]_{x_1=x_2=x_3}\\ &+\left(\frac{\partial}{\partial x_2}-\frac{\partial}{\partial x_3}\right)\left[\psi^{(Q_1\ldots 312\ldots Q_N)}-\psi^{(Q_1\ldots 213\ldots Q_N)}\right]_{x_1=x_2=x_3}\\ &+\left(\frac{\partial}{\partial x_3}-\frac{\partial}{\partial x_1}\right)\left[\psi^{(Q_1\ldots 123\ldots Q_N)}-\psi^{(Q_1\ldots 321\ldots Q_N)}\right]_{x_1=x_2=x_3}\\ &=4c(\mathcal{P}_\sigma^{12}+\mathcal{P}_\sigma^{23}+\mathcal{P}_\sigma^{31})\psi^{(Q_1\ldots 123\ldots Q_N)}\big|_{x_1=x_2=x_3}\end{aligned}\tag{11}$$

where we have set the three particles ijk to be 123. The $\psi^{(Q_1\ldots 123\ldots Q_N)}|_{x_1=x_2=x_3}$ on the right-hand side of the above relation can also be replaced by a wavefunction with another permutation of 123 according to the relation (10) of single-valuedness. On the left-hand side, it should be noted that the pairs of wavefunctions in the same bracket are not of neighbouring sectors as in (5).

We find the condition equation (11) can be fulfilled by the two-particle connection equation (5). As one can see, the relation (6) is independent of the particle coordinates, so it does not affect the cancellation of the amplitudes in (5) if we set a third adjacent particle k in the same position as the two particles on the barrier $x_i = x_j$ while keeping all the permutation notation Qs in the $A^{(Q)}$s unchanged. Hence, (5) can be extended to three-particle occupancy:

$$\begin{aligned}\left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\left[\psi^{(Q_1\ldots jik\ldots Q_N)}-\psi^{(Q_1\ldots ijk\ldots Q_N)}\right]&=2c\mathcal{P}_\sigma^{ij}\psi^{(Q_1\ldots ijk\ldots Q_N)}\\ \left(\frac{\partial}{\partial x_i}-\frac{\partial}{\partial x_j}\right)\left[\psi^{(Q_1\ldots kji\ldots Q_N)}-\psi^{(Q_1\ldots kij\ldots Q_N)}\right]&=2c\mathcal{P}_\sigma^{ij}\psi^{(Q_1\ldots kij\ldots Q_N)}\end{aligned}\tag{12}$$

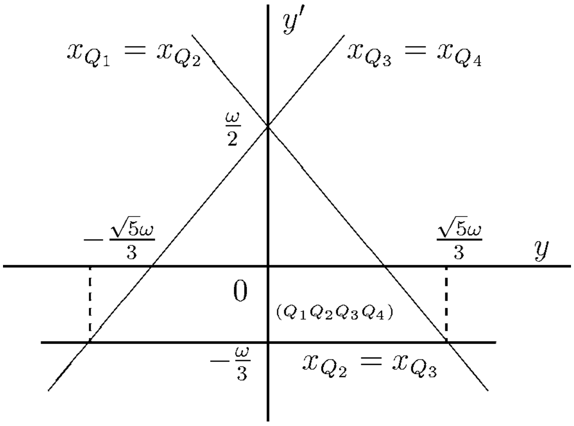


Figure 2. The region inside the triangle in the $y - y'$ plane with fixed ω belongs to sector $(Q_1Q_2Q_3Q_4)$, $\omega > 0$.

where $x_i = x_j = x_k$. Additionally the wavefunctions on the left-hand side are still defined on neighbouring regions as in (5). We split the differential operation $\partial/\partial x_1 - \partial/\partial x_2$ in (11) to be $(\partial/\partial x_1 - \partial/\partial x_3) + (\partial/\partial x_3 - \partial/\partial x_2)$ and similarly for the others; the left-hand side of (11) becomes

$$\begin{aligned}
&\left(\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right) + \left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}\right)\right)\left[\psi^{(Q_1\ldots 231\ldots Q_N)} - \psi^{(Q_1\ldots 132\ldots Q_N)}\right]_{x_1=x_2=x_3} \\
&\quad + \left(\left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right) + \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_3}\right)\right) \\
&\quad \times\left[\psi^{(Q_1\ldots 312\ldots Q_N)} - \psi^{(Q_1\ldots 213\ldots Q_N)}\right]_{x_1=x_2=x_3} \\
&\quad + \left(\left(\frac{\partial}{\partial x_3} - \frac{\partial}{\partial x_2}\right) + \left(\frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_1}\right)\right) \\
&\quad \times\left[\psi^{(Q_1\ldots 123\ldots Q_N)} - \psi^{(Q_1\ldots 321\ldots Q_N)}\right]_{x_1=x_2=x_3}.
\end{aligned} \tag{13}$$

In terms of (12), equation (13) is equal to $4c\,(\mathcal{P}_\sigma^{12} + \mathcal{P}_\sigma^{23} + \mathcal{P}_\sigma^{31})\psi^{(Q_1\ldots 123\ldots Q_N)}|_{x_1=x_2=x_3}$, i.e., exactly the right-hand side of (11). Therefore we have proved that the condition required by three-particle SPS is actually satisfied by the connection equation of two-particle scattering in the BA.

5. Four-electron scattering at the same position

Now let us turn to the case of four particles at the same point. For the sake of notation simplicity, we only consider four electrons: the N-electron case can be generalized straightforwardly. Similar to the analysis in three-electron scattering, it is sufficient to obtain the four-electron scattering condition by calculating the integral of (3) over an ϵ-sized Gaussian box with centre axis $x_1 = x_2 = x_3 = x_4$. A vector in the sector of $(Q_1Q_2Q_3Q_4)$ can be built on another set of orthonormal bases (refer to figure 2)

$$\begin{aligned}
\boldsymbol{r}_{Q_1Q_2Q_3Q_4} &\equiv x_{Q_1}\boldsymbol{e}_{Q_1} + x_{Q_2}\boldsymbol{e}_{Q_2} + x_{Q_3}\boldsymbol{e}_{Q_3} + x_{Q_4}\boldsymbol{e}_{Q_4} \\
&= \omega\boldsymbol{f}_{Q_1Q_2Q_3Q_4} + h\boldsymbol{\Gamma} + y\boldsymbol{g}_{Q_1Q_2Q_3Q_4} + y'\boldsymbol{g}'_{Q_1Q_2Q_3Q_4}
\end{aligned} \tag{14}$$

where the basis vectors are

$$
\begin{aligned}
g_{Q_1Q_2Q_3Q_4} &= \frac{e_{Q_1} - e_{Q_2} - e_{Q_3} + e_{Q_4}}{2} \qquad & g'_{Q_1Q_2Q_3Q_4} &= \frac{e_{Q_1} - 3e_{Q_2} + 3e_{Q_3} - e_{Q_4}}{\sqrt{20}} \\
f_{Q_1Q_2Q_3Q_4} &= -\frac{3e_{Q_1} + e_{Q_2} - e_{Q_3} - 3e_{Q_4}}{\sqrt{20}} \qquad & \Gamma &= \frac{e_1 + e_2 + e_3 + e_4}{2}
\end{aligned}
\tag{15}
$$

and Γ is the direction vector along axis $x_1 = x_2 = x_3 = x_4$. The boundary equations at $x_i = x_j$ can be expressed in terms of $f_{Q_1Q_2Q_3Q_4}$, Γ, $g_{Q_1Q_2Q_3Q_4}$ and $g'_{Q_1Q_2Q_3Q_4}$ according to the above relations. In the $(Q_1Q_2Q_3Q_4)$ sector of the Gaussian box the inwards normals of the intersecting hyperplanes $x_{Q_1} = x_{Q_2}$, $x_{Q_2} = x_{Q_3}$ and $x_{Q_3} = x_{Q_4}$ are respectively $n_{Q_2Q_1} = e_{Q_2} - e_{Q_1}$, $n_{Q_3Q_2} = e_{Q_3} - e_{Q_2}$ and $n_{Q_4Q_3} = e_{Q_4} - e_{Q_3}$. It is easy to see that $f_{Q_1Q_2Q_3Q_4} \cdot n_{21} = f_{Q_1Q_2Q_3Q_4} \cdot n_{32} = f_{Q_1Q_2Q_3Q_4} \cdot n_{Q_4Q_3}$. Setting $\omega = \epsilon$, we have a hyperplane which can be the side hypersurface of the Gaussian box in the $(Q_1Q_2Q_3Q_4)$ region; $f_{Q_1Q_2Q_3Q_4}$ is the outwards normal direction vector. We also set the size parameter ϵ to be infinitely small, carry out the expansion in the orders of ϵ and keep the lowest order in the integration of (3). After a careful calculation, we find the conditions required by four-particle SPS are

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{5}{18}\right) \sum_{Q\in S_4} &\left(3\frac{\partial}{\partial x_{Q_1}} + \frac{\partial}{\partial x_{Q_2}} - \frac{\partial}{\partial x_{Q_3}} - 3\frac{\partial}{\partial x_{Q_4}}\right) \psi^{(Q)}(x,x,x,x) \\
&= \frac{c}{2} \sum_{Q\in S_4} \left(\frac{5}{6}\mathcal{P}_\sigma^{Q_1Q_2} + \frac{10}{9}\mathcal{P}_\sigma^{Q_2Q_3} + \frac{5}{6}\mathcal{P}_\sigma^{Q_3Q_4}\right) \psi^{(Q)}(x,x,x,x).
\end{aligned}
\tag{16}
$$

Also the above four-particle interacting condition can be satisfied by the two-particle connection equation (5). Relations (12) are extended to

$$
\begin{aligned}
&\left(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\right)\left[\psi^{(Q_1\ldots jikl\ldots Q_N)} - \psi^{(Q_1\ldots ijkl\ldots Q_N)}\right] = 2c\mathcal{P}_\sigma^{ij}\psi^{(Q_1\ldots ijkl\ldots Q_N)} \\
&\left(\frac{\partial}{\partial x_j} - \frac{\partial}{\partial x_k}\right)\left[\psi^{(Q_1\ldots ikjl\ldots Q_N)} - \psi^{(Q_1\ldots ijkl\ldots Q_N)}\right] = 2c\mathcal{P}_\sigma^{jk}\psi^{(Q_1\ldots ijkl\ldots Q_N)} \\
&\left(\frac{\partial}{\partial x_k} - \frac{\partial}{\partial x_l}\right)\left[\psi^{(Q_1\ldots ijlk\ldots Q_N)} - \psi^{(Q_1\ldots ijkl\ldots Q_N)}\right] = 2c\mathcal{P}_\sigma^{kl}\psi^{(Q_1\ldots ijkl\ldots Q_N)}
\end{aligned}
\tag{17}
$$

where $x_i = x_j = x_k = x_l$. The left-hand side of (16) is rearranged by splitting the differential operators and using (17) respectively:

$$
\begin{aligned}
-\frac{1}{2}\left(\frac{5}{18}\right) \sum_{Q\in S_4} &\left[3\left(\frac{\partial}{\partial x_{Q_1}} - \frac{\partial}{\partial x_{Q_2}}\right) + 4\left(\frac{\partial}{\partial x_{Q_2}} - \frac{\partial}{\partial x_{Q_3}}\right)\right. \\
&\left. +3\left(\frac{\partial}{\partial x_{Q_3}} - \frac{\partial}{\partial x_{Q_4}}\right)\right]\psi^{Q}(x,x,x,x) \\
&= \frac{1}{4}\left(\frac{5}{18}\right) \sum_{Q\in S_4}\left[3\left(\frac{\partial}{\partial x_{Q_1}} - \frac{\partial}{\partial x_{Q_2}}\right)\left(\psi^{(Q_2Q_1Q_3Q_4)} - \psi^{(Q_1Q_2Q_3Q_4)}\right)\right. \\
&\quad +4\left(\frac{\partial}{\partial x_{Q_2}} - \frac{\partial}{\partial x_{Q_3}}\right)\left(\psi^{(Q_1Q_3Q_2Q_4)} - \psi^{(Q_1Q_2Q_3Q_4)}\right) \\
&\quad \left. +3\left(\frac{\partial}{\partial x_{Q_3}} - \frac{\partial}{\partial x_{Q_4}}\right)\left(\psi^{(Q_1Q_2Q_4Q_3)} - \psi^{(Q_1Q_2Q_3Q_4)}\right)\right] \\
&= c\sum_{Q\in S_4}\left[\tfrac{5}{12}\mathcal{P}_\sigma^{Q_1Q_2} + \tfrac{5}{9}\mathcal{P}_\sigma^{Q_2Q_3} + \tfrac{5}{12}\mathcal{P}_\sigma^{Q_3Q_4}\right]\psi^{(Q_1Q_2Q_3Q_4)}
\end{aligned}
\tag{18}
$$

which becomes precisely the right-hand side of equation (16). Thus we have proven that the four-electron SPS condition is satisfied by the conventional BA wavefunction.

6. Concluding remarks

In the above, we have worked out the conditions required by scatterings of three and four electrons at the same position and proved explicitly that they can be solved by the two-particle connection equations in the BA. The configurations with more than four particles at one spatial point are excluded from the present model by the Pauli principle: the antisymmetric wavefunctions vanish in such cases. Therefore the BA is valid for spin-exchange two-band electrons without the exclusion of multi-particle SPS. Unlike the degenerate Hubbard model, the BA wavefunction is confirmed to be the solution of the spatially continuous system and the definition of the BA wavefunction can be exactly extended to those cases of more than two particles occupying the same position.

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